University of Arkansas at little Rock
Department of Systems Engineering

SYEN 4399/5399 Estimation Theory
Fall 2008

Final EXAM
Thursday, December 11, 2008

- This is a closed book exam.
- Calculators are not allowed.
- There are 8 problems in the exam.
- The problems are not in order of difficulty. We recommend that you read through all the problems, then do the problems in whatever order suits you best.
- A correct answer does not guarantee full credit, and a wrong answer does not guarantee loss of credit. You should clearly but concisely indicate your reasoning and show all relevant work. Your grade on each problem will be based on our assessment of your level of understanding as reflected by what you have written in the space provided.
- Please be neat and box your final answer, we cannot grade what we cannot decipher.

Name

Problem 1

The heart rate h of a patient is automatically recorded by a computer every 100 ms. In 1 s the measurements $\hat{h}_1, \hat{h}_2, \cdots, \hat{h}_10$ are averaged to obtain $\hat{h}$. If $E(\hat{h}_i) = h$, and $\text{var}(\hat{h}_i) = 1$ for each i, determine the mean and variance of the average, i.e., determine $E(h)$ and $\text{var}(h)$. Does averaging improve the estimation? Assume each measurement is uncorrelated.

Problem 2: DC level in WGN

Consider a DC level in white Gaussian noise (WGN):

$$x[n] = A + w[n], \quad n = 0, 1, \cdots, N,$$

where $\mathrm{var}(w[n]) = \sigma^2$.
Determine the CRLB for A.

Problem 3: Fourier Analysis

Consider a data model consisting of sinusoids in white Gaussian noise:

$$x[n] = \sum_{k=1}^{M} a_k \cos(\frac{2\pi kn}{N}) + \sum_{k=1}^{M} b_k \sin(\frac{2\pi kn}{N}) + w[n], \quad n = 0, 1, \cdots, N, \tag{1}$$

where $w[n]$ is WGN. We want to estimate the amplitudes a_k, b_k of the cosines and sines. Therefore, the unknown vector parameter, θ, is given by

$$\theta = [a_1, a_2, \cdots, a_M, b_1, b_2, \cdots, b_M]^T.$$

1. Write Equation (1) in vector form, in terms of the linear model: $\mathbf{x} = \mathbf{H}\theta + \mathbf{w}$.
2. Find the MVU estimator of θ, $\hat{\theta}_{MVU}$.
3. The columns of the matrix $\mathbf{H}$ are orthogonal, so that $\mathbf{H}^T\mathbf{H} = \frac{N}{2}\mathbf{I}$. Simplify the expression of $\hat{\theta}_{MVU}$, and find the estimators $\hat{a}_k$ and $\hat{b}_k$.
4. Find the covariance matrix of $\hat{\theta}_{MVU}$, $\mathbf{C}_{\hat{\theta}}$.

Problem 4

Consider the following model:

$$x[n] = Ar^n + w[n], \quad n = 0, 1, \cdots, N, \tag{2}$$

where A is an unknown parameter, r is a known constant, and $w[n]$ is zero mean white (not necessarily Gaussian) with variance σ^2.

1. Write equation (2) in the linear model form.
2. Find the BLUE of A.
3. Find the minimum variance. Does it approach zero as $N \to \infty$?

Hint: Recall that $\hat{\theta}_{BLUE} = (\mathbf{H}^t\mathbf{C}^{-1}\mathbf{H})^{-1}\mathbf{H}^T\mathbf{C}^{-1}\mathbf{x}$, where $\mathbf{C}$ is the noise covariance matrix.

Problem 5

We observe N IID samples from an exponential distribution

$$p(x;\lambda) = \begin{cases} \lambda e^{-\lambda x}, & x > 0; \\ 0, & x < 0. \end{cases}$$

Find the MLE of the unknown parameter λ. Does this estimator make sense?

Problem 6

If the data set

$$x[n] = As[n] + w[n], \quad n = 0, 1, \cdots, N$$

is observed, where $s[n]$ is known and $w[n]$ is WGN with known variance σ^2, find the MLE of A.

Problem 7

In the below table, fill in the criterion of optimality and the statistical assumptions about the noise, for each estimator.

Model	Criterion of optimality	Assumptions about the noise	Solution
$\mathbf{x} = \mathbf{H}\theta + \mathbf{w}$			$\hat{\theta}_{MVU} = (\mathbf{H}^T\mathbf{H})^{-1}\mathbf{H}^T\mathbf{x}$
$\mathbf{x} = \mathbf{H}\theta + \mathbf{w}$			$\hat{\theta}_{BLUE} = (\mathbf{H}^T\mathbf{H})^{-1}\mathbf{H}^T\mathbf{x}$
$\mathbf{x} = \mathbf{H}\theta + \mathbf{w}$			$\hat{\theta}_{MLE} = (\mathbf{H}^T\mathbf{H})^{-1}\mathbf{H}^T\mathbf{x}$
$\mathbf{x} = \mathbf{H}\theta + \mathbf{e}$			$\hat{\theta}_{LS} = (\mathbf{H}^T\mathbf{H})^{-1}\mathbf{H}^T\mathbf{x}$

Table 1: Comparison of the MVU, BLUE, MLE and LS estimators

Problem 8

For the signal model

$$s[n] = \sum_{i=1}^{p} A_i \cos(2\pi f_i n) \tag{3}$$

where the frequencies f_i are known and the amplitudes A_i are to be estimated. Therefore, we have

$$\theta = \begin{pmatrix} A_1 \\ A_2 \\ \cdots \\ A_p \end{pmatrix}$$

1. Write model (3) in the linear form
2. Find the LS estimate of θ.